Bounds for the Castelnuovo-Mumford regularity of modules

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Abstract

We establish bounds for the Castelnuovo-Mumford regularity of a finitely generated graded module and its symmetric powers in terms of the degrees of the generators of the module and the degrees of their relations. We extend to modules (and improve) the known bounds for homogenous ideal in a polynomial ring established by Galligo, Guisti, Caviglia and Sbarra.

1. Introduction

Bayer and Stillman proved in [BS] that the complexity of an ideal (or a module) is the same as the one of its generic initial ideal. This connection motivated the search for bounds on the Castelnuovo-Mumford regularity in terms of the degrees of the generators of an ideal in order to control the complexity of Gröbner basis computations.

For an ideal I in a polynomial ring $R = k[X_1, \ldots, X_n]$, generated in degree at most d, Galligo ([Ga1], [Ga2]) and Giusti ([Giu]) proved the following bound if the characteristic of the field k is zero:

$$\mathrm{reg}(I) \leq (2d)^{2^{n-2}}.$$

A weaker bound was given in any characteristic by Bayer and Mumford ([BM]) following an argument of Mumford ([Mum]):

$$\mathrm{reg}(I) \leq (2d)^{(n-1)!}.$$

In [CS] Caviglia and Sbarra proved that the bound of Galligo and Giusti is valid in any characteristic. The example of Mayr and Meyer ([MM]) shows that the doubly exponential behavior in n cannot be avoided.

We extend the sharpest known bounds for ideals to finitely generated graded modules. As $\mathrm{reg}(I) = \mathrm{reg}(R/I) + 1$, the case of ideal corresponds to cyclic modules generated in degree zero and extending the result to arbitrary modules one should take into account the degrees of the generators and of their relations. Results on the regularity of modules have also been proved independently by Brodmann and Götsch in [BG]. The estimates they provide are compared to ours in Remark 3.7. Let us only point out that the exponent in our bound depends on the dimension of the support of the module which might be much smaller then the dimension of the ring appearing in their estimate.

We proceed in two steps. First we establish bounds in the case of modules supported in dimension at most 1. Second we extend the method of Caviglia and Sbarra to modules. It allows us to proceed by induction on the dimension of the module.

For the first step we use an argument first introduced by Gruson, Lazarsfeld and Peskine in proving regularity bounds for reduced curves: a complex which is not too far from being acyclic is enough to estimate the regularity. Our results are rather general as they hold for modules over each standard graded R_0-algebra where R_0 is an artinian local ring, provided the dimension of the

modules is at most 1, and for every module over a graded Cohen-Macaulay R_0-algebra in higher dimension. We establish the following bounds:

Theorem. *Let R be a standard graded Cohen-Macaulay algebra over the artinian local ring R_0 and let $M \neq 0$ be a graded R-module of finite type. Assume M is generated by n elements of non-negative degrees. Let c and δ be the codimension and the dimension of the support of M (so that $c + \delta = \dim R$), respectively. If M is generated in degrees at most $B-1$ and related in degrees at most B, then*:

(i) *If $\delta \leq 1$ and $c > 0$,* $\operatorname{reg}(M) \leq \operatorname{reg}(R) + (\dim R + n - 1)B - \dim R$.

(i)' *If $\delta \leq 1$ and $c = 0$,* $\operatorname{reg}(M) \leq \operatorname{reg}(R) + B - 1$.

(ii) *If $\delta \geq 2$ and $c > 0$,*

$$\operatorname{reg}(M) \leq \left[\deg(R)(\operatorname{reg}(R) + (c+n)B - c)\binom{c+n-1}{c}B^c\right]^{2^{\delta-2}}.$$

(ii)' *If $\delta \geq 2$ and $c = 0$,* $\operatorname{reg}(M) \leq [n \deg(R)(\operatorname{reg}(R) + B)]^{2^{\delta-2}}$.

This theorem can be used to improve the bound given by Caviglia and Sbarra in [CS] (see Remark 3.6).

It would be interesting to extend the bound in arbitrary dimension to the larger class of standard graded algebras over an artinian local ring. This could be achieved by extending Proposition 3.3 to this more general situation.

2. Regularity bounds for modules of dimension at most 1

Let $R = \oplus_{i \geq 0} R_i$ be a standard graded algebra over the artinian local ring $(R_0, \mathfrak{n})$, i.e. R_1 is a finite R_0-module and $R = R_0[R_1]$, and let M be a graded R-module with a finite presentation $F \xrightarrow{\varphi} G \longrightarrow M \longrightarrow 0$, where $F = \bigoplus_{i=1}^{m} R[-b_i]$, $G = \bigoplus_{i=1}^{n} R[-a_i]$ and $b_1 \geq b_2 \geq \ldots \geq b_m$.

Set $—^* := \operatorname{Hom}_R(—, R)$ and $\sigma := \sum_{i=1}^{n} a_i$. Let $E_\bullet^{(l)}$ be the complex of R-graded free R-modules associated to φ (see [E1]):

$$0 \longrightarrow N_{m-n}^{(l)}[\sigma] \xrightarrow{\delta} N_{m-n-1}^{(l)}[\sigma] \longrightarrow \ldots \longrightarrow N_l^{(l)}[\sigma] \xrightarrow{\varepsilon} L_l^{(l)} \xrightarrow{\nu} L_{l-1}^{(l)} \longrightarrow \ldots \longrightarrow L_0^{(l)} \longrightarrow 0$$

with $N_s^{(l)} = \operatorname{Sym}_{s-l} G^* \bigotimes \bigwedge^{n+s} F$ for $l \leq s \leq m-n$ and $L_s^{(l)} = \operatorname{Sym}_{l-s} G \bigotimes \bigwedge^{s} F$ for $0 \leq s \leq l$.

The complex $E_\bullet^{(l)}$ has graded degree zero differentials, its homology is supported in the support of M, $H_0(E_\bullet^{(l)}) = \operatorname{Sym}_R^l(M)$ for $l > 0$, and $H_0(E_\bullet^{(0)}) = \operatorname{Fitt}_0^R(M)$. The complexes $E_\bullet^{(0)}$ and $E_\bullet^{(1)}$ are known as Eagon-Northcott and Buchsbaum-Rim complexes, respectively.

We denote the local cohomology modules of M with support in the irrelevant maximal ideal $\mathfrak{m} = \oplus_{i>0} R_i$ by $H_{\mathfrak{m}}^i(M)$. Notice that $H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}+\mathfrak{n}}^i(M)$. We set

$$a_i(M) := \sup\{\mu \mid H_{\mathfrak{m}}^i(M)_\mu \neq 0\}.$$

Thus, $a_i(M) := -\infty$ if $H_{\mathfrak{m}}^i(M) = 0$. Recall that $\operatorname{reg}(M) = \max\{a_i(M) + i\}$.

We will provide a bound for the regularity of symmetric powers of M in terms of the degrees of the generators of M and of its first module of syzygies. We start by treating the case of modules of dimension at most 1. For simplicity, we state separately the result in case the ring has dimension at most one.

Proposition 2.0. *Let R be a standard graded algebra of dimension at most 1 over the artinian local ring R_0, M a graded R-module with a finite presentation $\bigoplus_{i=1}^m R[-b_i] \xrightarrow{\varphi} \bigoplus_{i=1}^n R[-a_i] \longrightarrow M \longrightarrow 0$. Assume that $b_1 \geq b_2 \geq \cdots \geq b_m$, set $b := \max\{b_i\} = b_1$ and $a := \max\{a_i\}$. Then, for $l > 0$,*

(i) If $\dim R = 0$, $\mathrm{reg}(R/\mathrm{Fitt}_R^0(M)) \leq \mathrm{reg}(R)$ and $\mathrm{reg}(\mathrm{Sym}_R^l(M)) \leq \mathrm{reg}(R) + la$,

(ii) If $\dim R = 1$,

$$\begin{aligned}\mathrm{reg}(\mathrm{Sym}_R^l(M)) &\leq \max\{a_0(R) + la, a_1(R) + (l-1)a + b\} \\ &\leq \mathrm{reg}(R) + \max\{la, (l-1)a + b - 1\},\end{aligned}$$

and

$$\mathrm{reg}(R/\mathrm{Fitt}_R^0(M)) \leq \mathrm{reg}(R) + \max\{0, \sum_{i=1}^n (b_i - a_i) - 1\}.$$

Over higher-dimensional rings we have:

Theorem 2.1. *Let R be a standard graded algebra of dimension $d \geq 2$ over the artinian local ring R_0, $M \neq 0$ a graded R-module of dimension at most 1 with a finite presentation $M = \mathrm{coker}(\bigoplus_{i=1}^m R[-b_i] \xrightarrow{\varphi} \bigoplus_{i=1}^n R[-a_i])$. (Note that we must have $m \geq n + d - 2$.)*

Assume that $b_1 \geq b_2 \geq \cdots \geq b_m$ and $a_1 \geq a_2 \geq \cdots \geq a_n$. Set $D_l := \sum_{i=1}^l (b_i - 1)$ and $\Delta := \sum_{i=1}^{\min\{m,n+d-1\}} b_i - \sum_{i=1}^n a_i - (d-1)a_n - d$.

Let $M' := \mathrm{coker}(\bigoplus_{i=1}^{n+d-2} R[-b_i] \xrightarrow{\varphi'} \bigoplus_{i=1}^n R[-a_i])$, where φ' is the restriction of φ.

Then,

(i) $\mathrm{reg}(R/\mathrm{Fitt}_R^0(M)) \leq \mathrm{reg}(R) + \Delta$,

(ii) for $l \leq d-1$, $\mathrm{reg}(\mathrm{Sym}_R^l(M)) \leq \begin{cases} \mathrm{reg}(R) + \max\{D_l, \Delta + la_n\} & \text{if } M \neq M' \text{ or } l \leq d-2 \\ \mathrm{reg}(R) + D_{d-1} & \text{if } M = M' \text{ and } l = d-1, \end{cases}$

(iii) for $l \geq d$, $\mathrm{reg}(\mathrm{Sym}_R^l(M)) \leq \mathrm{reg}(R) + D_d + (l-d)a_1$.

We will need the following lemma to prove the above results. It builds on ideas in [GLP] and generalizes Lemma 5.9 in [E2].

Lemma 2.2. *Let C be a complex of finite graded R-modules with $C_i = 0$ for $i < 0$. If $\dim(H_i(C)) \leq i$ for $i > 0$, then*

$$a_i(H_0(C)) \leq \max_{j \geq 0}\{a_{i+j}(C_j)\}, \quad \forall i.$$

In particular

$$\mathrm{reg}(H_0(C)) \leq \max_{0 \leq j \leq \dim R}\{\mathrm{reg}(C_j) - j\}.$$

Proof of Lemma 2.2. We consider the graded double complex $\mathcal{C}^\bullet_\mathfrak{m}C$ where $\mathfrak{m}$ is the maximal homogeneous ideal of R and $\mathcal{C}^\bullet_\mathfrak{m}E$ is the Čech complex on E. It gives rise to two spectral sequences. One of them has as second terms ${}'E_2^{pq} = H^p_\mathfrak{m}(H_q(C))$. Since $\dim(H_i(C)) \leq i$ for $i > 0$, ${}'E_2^{pq} = 0$ for $p > q > 0$. This implies that ${}'E_2^{p0} \simeq {}'E_\infty^{p0}$ for each p.

The other spectral sequence has as first terms, ${}''E_1^{pq} = H^p_\mathfrak{m}(C_q)$. It follows that $({}'E_\infty^{i0})_\mu \simeq ({}'E_2^{i0})_\mu = H^i_\mathfrak{m}(H_0(C))_\mu$ vanishes if $H^p_\mathfrak{m}(C_q)_\mu = 0$ for $p = q + i$. $\square$

Proof of Proposition 2.0:

If $\dim(R) = 0$, then $\mathrm{reg}(R/\mathrm{Fitt}^0_R(M)) \leq \mathrm{reg}(R)$ and, as $\mathrm{Sym}^l_R(M)$ is quotient of $\mathrm{Sym}^l_R(G)$,

$$\mathrm{reg}(\mathrm{Sym}^l_R(M)) \leq \mathrm{reg}(\mathrm{Sym}^l_R(G)) = \mathrm{reg}(R) + la_1.$$

If $\dim R = 1$, then by applying Lemma 2.2 to $E^{(l)}_\bullet$ we get, for $l = 0$ and $m \geq n$, $\mathrm{reg}(R/\mathrm{Fitt}^0_R(M)) \leq \max\{\mathrm{reg}(L_0^{(0)}), \mathrm{reg}(L_1^{(0)}) - 1\}$ and, for $l > 0$,

$$\mathrm{reg}(\mathrm{Sym}^l_R(M)) \leq \max\{a_0(L_0^{(l)}), a_1(L_1^{(l)}) - 1\} \leq \mathrm{reg}(R) + \max\{la, b + (l-1)a - 1\}.$$

The result follows. $\square$

Proof of Theorem 2.1:

Modifying the generators and relations of M, the modules $F := \bigoplus_{i=1}^m R[-b_i]$ and $G := \bigoplus_{i=1}^n R[-a_i]$ can be decomposed into $F = F_1 \oplus F_2 \oplus F_3$ and $G = G_1 \oplus G_2$ to obtain a presentation of M of the following type :

$$F_1 \oplus F_2 \oplus F_3 \xrightarrow{\begin{pmatrix} \psi & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} G_1 \oplus G_2$$

where ψ is a minimal presentation of M.

Notice that by passing to this minimal presentation, Δ and D_l can only decrease. It follows that we may assume that $F \xrightarrow{\varphi} G \longrightarrow M \longrightarrow 0$ is a minimal presentation of M. This assumption implies that $b_j > a_n$ for any j. Notice also that $b_j > a_1$ for $j = 1, \ldots, d-1$, because $M'' := \mathrm{coker}(F \xrightarrow{\varphi_1} R[-a_1])$ (where φ_1 is given by the first column of φ) is a quotient of M, hence its dimension is at most 1, which implies that at least $d-1$ of the degrees $b_i - a_1$ of the entries of φ_1 are positive.

The modules $H_s(E^l_\bullet)$ are supported in the support of M (see for instance [E2, A2.59]). As $\dim M \leq 1$ it implies that $\dim(H_s(E^l_\bullet)) \leq 1$ for all s. Therefore, by Lemma 2.2,

$$(*) \qquad \mathrm{reg}(H_0(E^l_\bullet)) \leq \max_{0 \leq s \leq d}\{\mathrm{reg}(E^l_s) - s\}.$$

The module $L_s^{(l)} = \mathrm{Sym}_{l-s}G \bigotimes \bigwedge^s F$ is a graded free R-module generated by elements of degrees $(a_{i_1} + \cdots + a_{i_{l-s}}) + (b_{j_1} + \cdots b_{j_s})$ with $i_1 \leq \cdots \leq i_{l-s}$ and $j_1 < \cdots < j_s$. Hence

$$\mathrm{reg}(L_s^{(l)}) = \mathrm{reg}(R) + \sum_{i=1}^s b_i + (l-s)a_1.$$

The module $N_s^{(l)} = \mathrm{Sym}_{s-l} G^* \bigotimes \bigwedge^{n+s} F$ is a graded free R-module generated by elements of degrees $-(a_{i_1} + \cdots + a_{i_{s-l}}) + (b_{j_1} + \cdots b_{j_{n+s}})$ with $i_1 \leq \cdots \leq i_{s-l}$ and $j_1 < \cdots < j_{n+s}$. Hence

$$\mathrm{reg}(N_s^{(l)}[\sigma]) = \mathrm{reg}(R) + \sum_{i=1}^{n+s} b_i + (l-s)a_n - \sigma.$$

Notice that, for $s \leq \min\{l, d-1\}$,

$$\mathrm{reg}(L_s^{(l)}) \leq \mathrm{reg}(R) + D_l + s,$$

because $b_j \geq a_1 + 1$ for $j \leq d-1$, and that

$$\mathrm{reg}(L_d^{(l)}) \leq \mathrm{reg}(R) + D_d + (l-d)a_1 + d.$$

In the case $M' \neq M$, one has $m \geq n+d-1$ and, for $l \leq s \leq d-1$,

$$\begin{aligned}
\mathrm{reg}(N_s^{(l)}[\sigma]) &= \mathrm{reg}(R) + \sum_{i=1}^{n+s} b_i + (l-s)a_n - \sigma \\
&= \mathrm{reg}(R) + \Delta + la_n - \sum_{i=n+s+1}^{n+d-1} b_i + (d-1-s)a_n + d \\
&\leq \mathrm{reg}(R) + \Delta + la_n - (d-1-s)(b_{n+d-1} - a_n) + d \\
&\leq \mathrm{reg}(R) + \Delta + la_n + s + 1
\end{aligned}$$

(because $b_j > a_n$ for any j).

A similar computation shows that $\mathrm{reg}(N_s^{(l)}[\sigma]) \leq \mathrm{reg}(R) + \Delta + la_n + s + 1$ for $s \leq m-n = d-2$ in the case $M = M'$ (recall that in this case $E_p^{(l)} = 0$ for $p \geq d$).

Inequality $(*)$ and the above estimates for $\mathrm{reg}(L_s^{(l)})$ and $\mathrm{reg}(N_s^{(l)}[\sigma])$ prove the inequalities stated in the Theorem. $\square$

Corollary 2.4. *Let R be a standard graded algebra over a field and let M be a graded R-module of dimension at most 1. Assume M is generated by n elements of degrees between 0 and $B-1$ and related in degrees at most B. If $\dim R > 0$ or $n > 1$, then*

$$\mathrm{reg}(M) \leq \mathrm{reg}(R) + (\dim R + n - 1)B - \dim R.$$

Proof. This immediately follows from Proposition 2.0 and Theorem 2.1 as $0 \leq a_i \leq B-1$ and $b_j \leq B$ for each i and j. $\square$

3. Regularity bounds for modules of arbitrary dimension

Let R be a standard graded algebra over the artinian local ring R_0 and let $M \neq 0$ be a graded R-module of dimension δ presented by $F = \bigoplus_{i=1}^{m} R[-b_i] \xrightarrow{\varphi} G = \bigoplus_{i=1}^{n} R[-a_i] \longrightarrow M \longrightarrow 0$.

We may write $R = S/J$, where J is a graded S-ideal and S is a polynomial ring over R_0. Assuming that J has no element of degree one, this presentation is unique and $S = \mathrm{Sym}_{R_0}(R_1)$.

Set

$$b_i^R(M) := \sup\{\mu \mid \mathrm{Tor}_i^R(M, R_0)_\mu \neq 0\}.$$

Thus, $b_i^R(M) = -\infty$ if $\mathrm{Tor}_i^R(M,k) = 0$. Recall that $\mathrm{reg}(M) = \max_i\{b_i^S(M) - i\}$. The initial degree of M is denoted by $\mathrm{indeg} M = \min\{\mu \mid M_\mu \neq 0\}$. Furthermore, we write $\lambda(M)$ for the length of M as R_0-module if it is finite, and we set $h^i_{\mathfrak{m}}(M)_\mu = \lambda(H^i_{\mathfrak{m}}(M)_\mu)$.

We will extend the technique of Caviglia and Sbarra to modules. We begin with a result that generalizes [CS, 2.2] :

Lemma 3.1. *Let M be a finitely generated graded R-module and let l be a linear form such that $K := 0 :_M (l)$ has finite length. Set $M' := M/H^0_{\mathfrak{m}}(M)$, $\overline{M} := M/lM$, $\overline{M'} := M'/lM'$, and $a := a_0(M) - \mathrm{indeg}(H^0_{\mathfrak{m}}(M)) + 1$.*

Then, for all integers μ,

(i) $\lambda(K_{\geq\mu}) = h^0_{\mathfrak{m}}(M)_\mu + h^0_{\mathfrak{m}}(\overline{M})_{>\mu} - h^0_{\mathfrak{m}}(\overline{M'})_{>\mu} \geq h^0_{\mathfrak{m}}(M)_\mu$,

(ii) $h^0_{\mathfrak{m}}(M)_{\mu+a} \leq \sum_{j=1}^a h^0_{\mathfrak{m}}(\overline{M})_{\mu+j} - \sum_{j=1}^a h^0_{\mathfrak{m}}(\overline{M'})_{\mu+j}$,

(iii) $\mathrm{reg}(M) \leq \mu - 1 + h^0_{\mathfrak{m}}(M)_\mu$, *provided* $\mu \geq \max\{b_0^R(M) + h - 1,\ b_1^R(M) - 1,\ \mathrm{reg}(\overline{M}) + 1\}$ *where* $h := \max\{b_0^S(J), 1\}$.

Proof. Consider the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^0_{\mathfrak{m}}(M) \arrow[r] \arrow[d, "\times l"] & M \arrow[r] \arrow[d, "\times l"] & M' \arrow[r] \arrow[d, "\times l"] & 0 \\
0 \arrow[r] & H^0_{\mathfrak{m}}(M)[1] \arrow[r] \arrow[d] & M[1] \arrow[r] \arrow[d] & M'[1] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0_{\mathfrak{m}}(M)/lH^0_{\mathfrak{m}}(M)[1] \arrow[r] \arrow[d] & \overline{M}[1] \arrow[r] \arrow[d] & \overline{M'}[1] \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

It shows that the length of K_μ is

$$\lambda(K_\mu) = h^0_{\mathfrak{m}}(M)_\mu - h^0_{\mathfrak{m}}(M)_{\mu+1} + h^0_{\mathfrak{m}}(\overline{M})_{\mu+1} - h^0_{\mathfrak{m}}(\overline{M'})_{\mu+1}$$

and the first equality follows. Set $F^j := 0 :_M (l^j)$ and notice that $F^0 = 0$ and $F^1 = K$. Using $lF^j = F^{j-1} \cap lM$ if $j \geq 2$, we get the exact sequence

$$0 \longrightarrow F^j/F^{j-1} \xrightarrow{\times l} F^{j-1}/F^{j-2}[1] \longrightarrow (F^{j-1} + lM)/(F^{j-2} + lM)[1] \longrightarrow 0.$$

It provides that

$$\lambda(F^j_\mu/F^{j-1}_\mu) = \lambda(F^{j-1}_{\mu+1}/F^{j-2}_{\mu+1}) - \lambda((F^{j-1} + lM)_{\mu+1}/(F^{j-2} + lM)_{\mu+1}),$$

thus in particular, $\lambda(F_\mu^j/F_\mu^{j-1}) \le \lambda(K_{\mu+j-1})$ for all $j \ge 1$. As $F_j = F_{j+1} = H^0_{\mathfrak m}(M)$ if $j \ge a$, it follows that

$$\begin{aligned} h^0_{\mathfrak m}(M)_\mu &= \sum_{j=1}^{a} \lambda(F_\mu^j/F_\mu^{j-1}) \\ &\le \sum_{j=1}^{a} \lambda(K_{\mu+j-1}) \\ &\le h^0_{\mathfrak m}(M)_\mu - h^0_{\mathfrak m}(M)_{\mu+a} + \sum_{j=1}^{a} h^0_{\mathfrak m}(\overline{M})_{\mu+j} - \sum_{j=1}^{a} h^0_{\mathfrak m}(\overline{M'})_{\mu+j}, \end{aligned}$$

which proves the second inequality.

For (iii), notice that $\operatorname{reg}(M) = \min\{a_0(M), \operatorname{reg}(\overline{M})\}$. Suppose we know for some $\mu \ge \operatorname{reg}(\overline{M})$ that $h^0_{\mathfrak m}(M)_j > h^0_{\mathfrak m}(M)_{j+1}$ whenever $j \ge \mu$ and $h^0_{\mathfrak m}(M)_j \ne 0$. Then it follows that $a_0(M) \le \mu - 1 + h^0_{\mathfrak m}(M)_\mu$, thus $\operatorname{reg}(M) \le \mu - 1 + h^0_{\mathfrak m}(M)_\mu$.

If $\mu \ge \operatorname{reg}(\overline{M})$ we have the exact sequence $0 \longrightarrow K_\mu \longrightarrow H^0_{\mathfrak m}(M)_\mu \longrightarrow H^0_{\mathfrak m}(M)_{\mu+1} \longrightarrow 0$. It implies that it suffices to estimate the degrees of the generators of K as S-module. Now the exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow M[1] \longrightarrow \overline{M}[1] \longrightarrow 0$ provides

$$b_0^S(K) \le \max\{b_0^S(M),\ b_1^S(M) - 1,\ b_2^S(\overline{M}) - 1\}.$$

We get a free resolution of M as S-module from the double complex obtained by taking a free resolution of M as R-module and then resolving each of the occurring free R-modules over S. This implies in particular that $b_1^S(M) \le \max\{b_1^R(M),\ b_0^R(M) + b_0^S(J)\}$. It follows that

$$b_0^S(K) \le \max\{b_0^R(M) + h - 1,\ b_1^R(M) - 1,\ \operatorname{reg}(\overline{M}) + 1\}.$$

Now the last statement follows. $\square$

Lemma 3.1 allows us to establish the following recursion:

Lemma 3.2. *Let $l_1, \ldots, l_{s+1} \in R$ be linear forms and set $M_i := M/(l_1, \ldots, l_i)M$, $i = 0, \ldots, s+1$. Assume that, for each i, $K_i := \ker(M_i \xrightarrow{\times l_{i+1}} M_i[1])$ has finite length and that M is generated in non-negative degrees.*

Set $Q_i = \max\{\operatorname{reg}(M_i), \lambda(K_i), b_1^R(M) - 2, b_0^R(M) + \max\{1, b_0^S(J)\} - 2\} + 1$ for $0 \le i \le s$.

Then, for each $i = 0, \ldots, s-1$, $Q_i \le Q_{i+1}^2$. In particular,

$$\operatorname{reg}(M) \le Q_s^{2^s}.$$

Proof. Lemma 3.1 (i) provides $h^0_{\mathfrak m}(M_i)_\mu \le \lambda((K_i)_{\ge \mu}) \le \lambda(K_i)$. Since $H^0_{\mathfrak m}(M_i) \subset M_i$ does not have elements of negative degree, we get

$$h^0_{\mathfrak m}(M_i) := \lambda(H^0_{\mathfrak m}(M_i)) \le (a_0(M_i) + 1) \cdot \lambda(K_i).$$

Define $r_i := \max\{b_1^R(M) - 2,\ b_0^R(M) + \max\{1, b_0^S(J)\} - 2,\ \operatorname{reg}(M_i)\}$. Then $Q_i = 1 + \max\{r_i, \lambda(K_i)\}$. Furthermore, set $R_{(i)} = R/(l_1, \ldots, l_i)R$. Using $b_0^{R_{(i)}}(M_i) \le b_0^R(M)$ and $b_1^{R_{(i)}}(M_i) \le b_1^R(M)$ and applying Lemma 3.1 (iii) to the $R_{(i)}$-module M_i, we obtain:

$$\begin{aligned} \operatorname{reg}(M_i) &\le r_{i+1} + h^0_{\mathfrak m}(M_i)_{r_{i+1}+1} \\ &\le r_{i+1} + \lambda((K_i)_{> r_{i+1}}) \\ &\le r_{i+1} + \lambda(K_i). \end{aligned}$$

In particular, this implies

$$r_i \leq r_{i+1} + \lambda(K_i).$$

Using the first inequality obtained above, we conclude that

$$(**) \qquad \lambda(K_i) \leq h^0_{\mathfrak{m}}(M_{i+1}) \leq (a_0(M_{i+1}) + 1) \cdot \lambda(K_{i+1}) \leq (r_{i+1} + 1) \cdot \lambda(K_{i+1}).$$

It follows that $\lambda(K_i) \leq Q_{i+1}(Q_{i+1} - 1)$ and $r_i \leq (Q_{i+1} - 1) + Q_{i+1}(Q_{i+1} - 1) = Q^2_{i+1} - 1$, thus $Q_i \leq Q^2_{i+1}$, as claimed. $\square$

To apply the previous lemma, we need to estimate the degree (or multiplicity) of a graded module in terms of the degrees appearing in a graded presentation of it.

Proposition 3.3. *Let R be a standard graded Cohen-Macaulay ring and let M be a graded R-module of codimension $c > 0$ that is generated by n elements of degrees $a_1, \ldots, a_n$ and whose first syzygy module is generated in degrees $b_1 \geq \ldots \geq b_s$. Then, $s \geq c + n - 1$ and*

$$\deg(M) \leq \deg(R) \sum_{1 \leq i_1 \leq \cdots \leq i_c \leq n} \prod_{\ell=1}^{c} (b_{i_\ell + \ell - 1} - a_{i_\ell}).$$

Proof. Replacing the generators $g_1, \ldots, g_s$ of the first syzygy module by $g'_i := g_i + \sum_{j>i} h_{ij} g_j$, where h_{ij} is a sufficiently general polynomial of degree $b_i - b_j$, we may assume that the $(c + n - 1) \times n$ matrix H corresponding to the relations $g_1, \ldots, g_{c+n-1}$ has its ideal of maximal minors of codimension c. Since M is a quotient of the module P, this implies $\deg(M) \leq \deg(P)$. We will now compute the degree of P.

In case R is a polynomial ring over a field k, [Fu, 14.4.1] shows that the degree of P is equal (up to the sign) to the coefficient of order c in the expansion of

$$\frac{\prod_{i=1}^{s}(1 - b_i t)}{\prod_{j=1}^{n}(1 - a_j t)}.$$

Now, setting σ_p for the p-th symmetric function in s variables, and m_q for the sum of monomials of degree q in n variables, one has:

$$\prod_{i=1}^{s}(1 - b_i t) = \sum_{p=0}^{s} (-1)^p \sigma_p(b_1, \ldots, b_s) t^p$$

and

$$\frac{1}{\prod_{j=1}^{n}(1 - a_j t)} = \sum_{q \geq 0} m_q(a_1, \ldots, a_n) t^q.$$

It follows that $\deg(P) = \sum_{p+q=c} (-1)^{p-c} \sigma_p(b_1, \ldots, b_s) m_q(a_1, \ldots, a_n)$, which can be rewritten as

$$\sum_{1 \leq i_1 \leq \cdots \leq i_c \leq n} \prod_{\ell=1}^{c} (b_{i_\ell + \ell - 1} - a_{i_\ell}).$$

Set $a := (a_1, \ldots, a_n)$, $b := (b_1, \ldots, b_{c+n-1})$ and let $H_{a,b}(\mu)$ be the Euler-Poincaré characteristic of the Buchsbaum-Rim complex $(E^{(1)}_\bullet)_\mu$. Since P is resolved by $E^{(1)}_\bullet$, there is polynomial $P_{a,b}(t)$ such that

$$\sum_{\mu} H_{a,b}(\mu) t^\mu = \frac{P_{a,b}(t)}{(1-t)^{p-c}} = (1-t)^c P_{a,b}(t) \frac{1}{(1-t)^p} = (1-t)^c P_{a,b}(t) \frac{P_R(t)}{(1-t)^p},$$

where $p := \dim(R)$, and $\sum_{1\leq i_1\leq\cdots\leq i_c\leq n}\prod_{\ell=1}^{c}(b_{i_\ell+\ell-1}-a_{i_\ell}) = \deg(P) = P_{a,b}(1)$. The same computations give the analogous result if R is a polynomial ring over an artinian local ring R_0.

If R is not a polynomial ring over R_0, then the Buchsbaum-Rim complex is still a resolution of M. Since the shifts occurring in the modules $E_i^{(1)}$ are given by the same expressions in terms of a and b as above, the Euler-Poincaré characteristics $H'_{a,b}(\mu)$ of $(E_\bullet^{(1)})_\mu$ of degree μ is given by

$$\sum_\mu H'_{a,b}(\mu)t^\mu = (1-t)^c P_{a,b}(t)\frac{P_R(t)}{(1-t)^p}$$

where $p := \dim R$ and $P_R(1) = \deg(R)$. The conclusion follows. $\square$

Corollary 3.4. *With the hypotheses of Proposition 3.3, set $a := \min_i\{a_i\}$. Then*

$$\deg(M) \leq \deg(R)\binom{r+n-1}{n-1}\prod_{i=1}^{r}(b_i-a).$$

Theorem 3.5. *Let R be a standard graded Cohen-Macaulay ring over the artinian local ring R_0 and let $M \neq 0$ be graded R-module of finite type. Assume M is generated by n elements of non negative degrees. Let c and δ be the codimension and the dimension of the support of M (so that $c+\delta = \dim R$), respectively. If M is generated in degrees at most $B-1$ and related in degrees at most B, then*:

(i) *If $\delta \leq 1$ and $c > 0$,* $\operatorname{reg}(M) \leq \operatorname{reg}(R) + (\dim R + n - 1)B - \dim R$.

(i)' *If $\delta \leq 1$ and $c = 0$,* $\operatorname{reg}(M) \leq \operatorname{reg}(R) + B - 1$.

(ii) *If $\delta \geq 2$ and $c > 0$,*

$$\operatorname{reg}(M) \leq \left[\deg(R)(\operatorname{reg}(R) + (c+n)B - c)\binom{c+n-1}{c}B^c\right]^{2^{\delta-2}}.$$

(ii)' *If $\delta \geq 2$ and $c = 0$,*

$$\operatorname{reg}(M) \leq [n\deg(R)(\operatorname{reg}(R)+B)]^{2^{\delta-2}}.$$

Proof. Parts (i) and (i)' are proved in Theorem 2.1 and Proposition 2.0.

Let $D := \deg(R)$ and $r := \operatorname{reg}(R)$. We may assume that the field $R_0/\mathfrak{n}$ is infinite.

If $\dim M = 1$, one has $\lambda(K) = \lambda(M/lM) - \deg(M)$ where $l \in R$ is any linear form such that $K := 0 :_M (l)$ has finite length. Corollary 3.4 applied to M/lM provides $\lambda(M/lM) \leq D\binom{c+n-1}{n-1}B^c$ if $c > 0$. Notice that $\lambda(M/lM) \leq nD$ if $c = 0$. It follows that $\lambda(K) < D\binom{c+n-1}{n-1}B^c$ if $\dim M = 1$ and $c > 0$ and $\lambda(M/lM) \leq nD$ if $\dim M = 1$ and $c = 0$.

Let $\delta = 2$ and $c > 0$ and choose l as a general element in R_1. Notice that $b_0^R(M) + \max\{1, b_0^S(J)\} \leq (B-1)+(r+1)$. Hence, using the notation of Lemma 3.2, part (i) implies

$$r_1 = \max\{b_1^R(M)-2, b_0^R(M)+\max\{1,b_0^S(J)\}-2, \operatorname{reg}(M/lM)\} \leq r + (c+n)B - (c+1).$$

Thus we get from Lemma 3.1 (i) and (iii) that $\operatorname{reg}(M) < \lambda(K) + r + (c+n)B - c$. Moreover, inequality $(**)$ in the proof of Lemma 3.2 provides

$$\lambda(K) \leq (r+(c+n)B-c)\left[D\binom{c+n-1}{n-1}B^c - 1\right].$$

It follows that

$$\begin{aligned}\max\{\operatorname{reg}(M),\lambda(K)\}+1 &\le (r+(c+n)B-c)\left[D\binom{c+n-1}{n-1}B^c-1\right]+r+(c+n)B-c\\ &= D(r+(c+n)B-c)\binom{c+n-1}{n-1}B^c.\end{aligned}$$

If $\delta = 2$ and $c = 0$, the estimates are respectively $r_1 \le r + B - 1$ by part (i)', $\operatorname{reg}(M) < \lambda(K) + r + B$ by Lemma 3.1 (i) and (iii) and $\lambda(K) \le (r + B)(nD - 1)$ by inequality (∗∗) in the proof of Lemma 3.2. It follows that $\max\{\operatorname{reg}(M), \lambda(K)\} + 1 \le nD(r + B)$ in this case.

Finally, if $\delta > 2$, the result follows by induction using Lemma 3.2 with $s = \delta - 2$. □

Example 3.6. Consider the special case where $R = S$ is a polynomial ring over an artinian local ring R_0 and $M = S/I$ is of codimension c, dimension δ, and I is generated in degree at most B. Then Theorem 3.5 provides

$$\operatorname{reg}(I) \le \left[(c+1)B^{c+1}\right]^{2^{\delta-2}}.$$

Notice that if $\dim S = p \ge 2$ and $c = 1$, then $I = (F)J$, where F has degree $e \le B$ and the ideal J is generated in degree at most $B - e$ and has codimension $c' \ge 2$. Then we get

$$\operatorname{reg}(I) = e + \operatorname{reg}(J) \le e + \left[(c'+1)(B-e)^{c'+1}\right]^{2^{p-4}}.$$

It follows that the regularity of every ideal in $I \subset S := R_0[X_1, \dots, X_p]$ that is generated in degree at most B is bounded above by $p(B - 1) + 1$ if $p \le 3$, and by $\left[3B^3\right]^{2^{p-4}}$ if $p \ge 4$. Notice that these results also follow from [Ch1, 3.3] or [Sj, Theorem 2] and [CF, Theorem 1] by using Lemma 3.2. In fact, it gives the slightly refined bound (which also follows from the proof of Theorem 3.5):

$$\operatorname{reg}(I) \le \left[3B^2(B-1)\right]^{2^{p-4}} + 1, \quad \text{provided } p \ge 4.$$

This in turn improves the bound of Caviglia and Sbarra [CS]: $\operatorname{reg}(I) \le \left[B^2 + 2B - 1\right]^{2^{p-3}}$.

Remark 3.7.(i) In part (iii) of Theorem 3.5 the same arguments show a refined inequality if $c > 0$. Namely, if M is generated in degrees $a_1, \dots, a_s$ and related in degrees $b_1 \ge \dots \ge b_s$, then (unless $s = c+n-1$ in which case M is Cohen-Macaulay of regularity $\operatorname{reg}(R)+\sum_j b_j - \sum_i a_i - c\min\{a_i\}$) one has

$$\operatorname{reg}(M) \le \left[\deg(R)(\operatorname{reg}(R) + b_1 + \cdots + b_{c+n} - c) \sum_{1\le i_1\le\cdots\le i_r\le n} \prod_{\ell=1}^{r} (b_{i_\ell+\ell-1} - a_{i_\ell})\right]^{2^{\delta-2}}.$$

(ii) In [BG, 6.3], Brodmann and Götsch prove the following bound:

$$\operatorname{reg}(M) \le [\operatorname{reg}(R) + (n+1)\deg(R) + B + 1]^{2^{p-1}},$$

where $p = \dim R$. In fact, they give a bound that is a little more precise (using more data on the a_i's and b_j's). The main difference to our bound is that in our estimate the exponent depends on

the dimension of the module as opposed to the dimension of the ambient ring. Furthermore, our bound is slightly stronger, even in the case of a module supported in small codimension.

Remark 3.8. If M is generated in degrees between 0 and a and related in degrees at most B, then $\mathrm{Sym}^l_R(M)$ is generated in degrees between 0 and la and related in degrees at most $B+(l-1)a$. Applying Theorem 3.5 in this situation, one obtains if $a \leq B-1$ and $\delta = \dim(M) \geq 2$,

$$\mathrm{reg}(\mathrm{Sym}^l_R(M)) \leq \left[\deg(R)(\mathrm{reg}(R) + (c+n)(B+(l-1)a-c)\binom{c+n-1}{c}(B+(l-1)a)^c\right]^{2^{\delta-2}}.$$

Remark 3.9. It would be interesting to extend the bound in Theorem 3.5 to the class of arbitrary standard graded algebras over an artinian local ring.

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